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Winning sets, quasiconformal maps and Diophantine approximation

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Abstract

This paper describes two new types of winning sets in $\mathbb{R}^n$, defined using variants of Schmidt's game. These *strong* and *absolute* winning sets include many Diophantine sets of measure zero and first category, and have good behavior under countable intersections. Most notably, they are invariant under quasiconformal maps, while classical winning sets are not.

Contents

1 Introduction

Winning sets. Suppose two players, A and B, take turns choosing a nested sequence of closed Euclidean balls

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset B_3 \cdots$$

in $\mathbb{R}^n$ whose diameters satisfy, for fixed $0 < \alpha, \beta < 1$,

$$|A_i| = \alpha |B_i| \quad \text{and} \quad |B_{i+1}| = \beta |A_i|. \tag{1.1}$$

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Following Schmidt [Sch], we say $E \subset \mathbb{R}^n$ is an (α, β)-*winning set* if player A has a strategy which guarantees that $\bigcap A_i$ meets E. We say E is an α-*winning set* if it is (α, β)-winning for all $0 < \beta < 1$, and a *winning set* if it is α-winning for some $\alpha > 0$.

Winning sets have many useful properties; for example:

1. Any winning set in $\mathbb{R}^n$ has Hausdorff dimension n.
2. A countable intersection of α-winning sets is α-winning.
3. Winning sets are preserved by bi-Lipschitz homeomorphisms of $\mathbb{R}^n$.

See [Sch] and [Dani3, Prop 5.3]. However, as we will see in §4:

Theorem 1.1 *Winning sets are generally not preserved by quasisymmetric maps.*

Here a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is k-*quasisymmetric* if for any ball $B(x, r) \subset \mathbb{R}^n$, there exists an $s > 0$ such that

$$B(\phi(x), s) \subset \phi(B(x, r)) \subset B(\phi(x), ks). \tag{1.2}$$

In dimensions $n \geq 2$, quasisymmetric maps are the same as quasiconformal maps [Geh2, Thm 7.7].

Strong winning. In this note, we propose two variants of Schmidt's game. In the first, (1.1) is replaced by the pair of inequalities:

$$|A_i| \geq \alpha |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |A_i|. \tag{1.3}$$

If A has a winning strategy in the game defined by (1.3), we say E is an (α, β)-*strong winning set.* Sets which are α-strong winning and strong winning are defined analogously.

It is straightforward to verify that:

1. A strong winning set $E \subset \mathbb{R}^n$ is winning; in particular, $\mathrm{H.dim}(E) = n$; and
2. A countable intersection of α-strong winning sets is α-strong winning.

In addition, property (3) for winning sets can be strengthened to:

Theorem 1.2 *Strong winning sets are preserved by quasisymmetric homeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$.*

See §2.

Absolute winning. As a second variant of Schmidt's game, suppose A and B choose a sequence of balls A_i and B_i such that the sets

$$B_1 \supset (B_1 - A_1) \supset B_2 \supset (B_2 - A_2) \supset B_3 \ldots \tag{1.4}$$

are nested, and for fixed $0 < \beta < 1/3$, we have

$$|B_{i+1}| \geq \beta |B_i| \quad \text{and} \quad |A_i| \leq \beta |B_i|. \tag{1.5}$$

We say E is an *absolute* winning set if for all $\beta \in (0, 1/3)$, player A has a strategy which guarantees that $\bigcap B_i$ meets E. (Note that in this game, player A has rather little control over $\bigcap B_i$: essentially, he can just block one of B's possible moves at the next step.)

It is straightforward to see that:

1. An absolute winning set is α-strong winning for all $\alpha \in (0, 1/2)$; and
2. A countable intersection of absolute winning sets is absolutely winning.

Moreover, the proof of Theorem 1.2 can be adapted to show (see §2):

3. Absolute winning sets are preserved by quasisymmetric homeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$.

In particular, (1) and (3) show the 'strength' α of an absolute winning set cannot be degraded by taking its quasisymmetric image.

Example. The set of real numbers with infinitely many zeros in their base 10 representation is strong winning but not absolutely winning; cf. [Sch, Theorem 5].

Diophantine sets. Now let $\Gamma \subset \text{Isom}(\mathbb{H}^{n+1})$ be a lattice in the isometry group of hyperbolic space, with a cusp at infinity. Then the endpoints of lifts of all bounded geodesic rays in $\mathbb{H}^{n+1}/\Gamma$ determine a *Diophantine set*

$$D(\Gamma) \subset \mathbb{R}^n \subset \partial \mathbb{H}^{n+1}.$$

For example,

$$D(\text{SL}_2(\mathbb{Z})) = \left\{ x \in \mathbb{R} \; : \; \exists \, C > 0 \; : \; \left| x - \frac{p}{q} \right| > \frac{C}{q^2} \text{ for all } p/q \in \mathbb{Q} \right\}$$

consists of the real numbers which are badly approximable by rationals. In general, $D(\Gamma)$ consists of the points in $\mathbb{R}^n$ which are badly approximable by the cusps of Γ (see §3).

It is known that $D(\Gamma)$ is a winning set [Dani1] (see [Sch] for the case of $\text{SL}_2(\mathbb{Z})$). Sharpening this result, in §3 we give a geometric proof of:

Theorem 1.3 *For any lattice $\Gamma \subset \text{Isom}(\mathbb{H}^{n+1})$ with a cusp at infinity, the Diophantine set $D(\Gamma)$ is absolutely winning.*

Corollary 1.4 *Let $D = D(\text{SL}_2(\mathbb{Z}))$ be the set of badly approximable numbers. Then for any sequence of k_i-quasisymmetric maps $\phi_i : \mathbb{R} \to \mathbb{R}$, the set $\bigcap \phi_i(D)$ has Hausdorff dimension one (but measure zero).*

(In fact $\bigcap \phi_i(D)$ is absolutely winning.) Note that we allow $k_i \to \infty$.

Example: Fuchsian groups. To motivate these results, consider a homeomorphism $h : X_1 \to X_2$ between a pair of finite-volume hyperbolic Riemann surfaces, each with a unique cusp. Their universal covers can be normalized so that $X_i = \mathbb{H}/\Gamma_i$, $i = 1, 2$ and Γ_i has a cusp at $z = \infty$. Then h can be lifted to a homeomorphism $\widetilde{h} : \mathbb{H} \to \mathbb{H}$ whose boundary values give a quasisymmetric map

$$\phi : \mathbb{R} \to \mathbb{R}.$$

The map ϕ depends only on the isotopy class of h, and is compatible with an isomorphism $\Gamma_1 \cong \Gamma_2$.

Now assume h is not isotopic to an isometry. Then ϕ is highly singular: it does not preserve sets of measure zero, and it even sends a set of full measure to a set of Hausdorff dimension < 1 (cf. [Ku], [Mc]). Nevertheless, the bounded geodesic rays in X_1 and X_2 determine a pair of Diophantine sets D_1 and D_2 of Hausdorff dimension one satisfying $\phi(D_1) = D_2$. Thus it is natural to ask if $\mathrm{H.dim}\,\phi(D_1) = 1$ for all quasisymmetric ϕ. A positive answer is provided by Theorems 1.2 and 1.3.

Remark: Porous sets. A set $E \subset \mathbb{R}^n$ is *porous* if there exists a $\delta > 0$ such that for all $x \in \mathbb{R}^n$ and $1 > r > 0$, $B(x, r)$ contains a ball of the form $B(y, \delta r)$ disjoint from E. A countable union of porous sets is *σ-porous.* These sets enjoy properties parallel to those of strong winning sets; for example, one can readily verify [Vai2]:

1. A σ-porous set in $\mathbb{R}^n$ has measure zero and is meager in the sense of Baire category.
2. A countable union of σ-porous sets is σ-porous.
3. Quasisymmetric mappings preserve σ-porous sets.

In §3 we will also see:

Proposition 1.5 *For any lattice with a cusp at infinity, $D(\Gamma)$ is σ-porous.*

Thus the Diophantine set $D(\Gamma)$ is simultaneously large (in the sense of games and Hausdorff dimension) and small (in the sense of measure, category and porosity), and both properties are preserved under quasisymmetric maps. (This explains the measure zero remark in Corollary 1.4.)

For additional perspective, in §3 we define a *steering game* in which A and B take turns constructing a quasigeodesic ray $\gamma \subset \mathbb{H}^{n+1}$. Then player A can guide the endpoint of γ into $E \subset \partial\mathbb{H}^{n+1}$ iff E is a strong winning set.

Question. Do the bounded Teichmüller geodesics in $\mathcal{M}_{g,n}$ determine a strong winning set in $\mathbb{P}\mathcal{ML}_{g,n}$?

Notes and references. Winning sets arise naturally in many other settings in dynamics and Diophantine approximation (see e.g. [Dani2], [Ts], [Fi], [KW]); it seems likely that most of these sets are also strong winning. See [BBFKW, Cor. 4.2] for a result related to Corollary 1.4 above (but involving bilipschitz maps). The modified rule (1.3) is also mentioned at the end of [Fa].

Schmidt's game is a constrained form of the Banach–Mazur game, which is discussed in [Ox] and [Te]. Another counterexample for Schmidt's game is given in [Fr].

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Notation. The expression $A \asymp B$ means $0 < A/C < B < CA$ for an implicit constant C. The closed ball of radius r about x is denoted $B(x,r)$, and $B(E,r) = \bigcup_{x\in E} B(x,r)$ for any set E. The diameter of a set is given by $|E| = \sup\{d(x,y) \,:\, x,y \in E\}$.

We use $\mathbb{N}$ to denote the naturals numbers $\{0,1,2,\ldots\}$. The decomposition of a real number into its integral and fractional parts is given by $x = \lfloor x \rfloor + (x \bmod 1)$; here $\lfloor x \rfloor \in \mathbb{Z}$ and $(x \bmod 1) \in [0,1)$.

2 Transport of strategies

In this section we prove the following uniform version of Theorem 1.2.

Theorem 2.1 *The image of an α-strong winning set $E \subset \mathbb{R}^n$ under a k-quasisymmetric map ϕ is α'-strong winning, where α' depends only on (α,k,n).*

By similar reasoning we will show:

Theorem 2.2 *Absolute winning sets are preserved by quasisymmetric homeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$.*

Quasisymmetric maps. The proof will use the following standard properties of k-quasisymmetric maps $\phi : \mathbb{R}^n \to \mathbb{R}^n$.

1. For any ball $B \subset \mathbb{R}^n$ there exists a ball $B' \subset \phi(B)$ such that
$$|B'| \asymp |\phi(B)|.$$
2. For any sets $A \subset B \subset \mathbb{R}^n$ with $|A| \geq |B|/2$, we have $|\phi(A)| \asymp |\phi(B)|$.

3. More generally, if $A \subset B$ then

$$\frac{|\phi(A)|}{|\phi(B)|} > \xi_{k,n}\left(\frac{|A|}{|B|}\right) > 0. \tag{2.1}$$

4. The same bounds hold for ϕ^{-1}.

Here the implicit constants and the function $\xi_{k,n}$ depend only on (k, n).

Property (1) is immediate from the definition of quasisymmetry (equation (1.2)), but properties (2), (3) and (4) (for $n > 1$) are not. They can be established by first showing ϕ and ϕ^{-1} are k-quasiconformal, and then applying distortion estimates for ring domains; see e.g. [Geh2], [Vai1, §18], [AVV, Ch. 14]. Note that property (3) follows by iterating property (2).

Strategies. A *positional strategy* for Schmidt's game or its variants is a function f from the set of balls in $\mathbb{R}^n$ to itself satisfying $f(B) \subset B$. To use this strategy, player A makes the sequence of moves $A_i = f(B_i)$. It can be shown that whenever player A can win, he can do so using a positional strategy [Sch, Thm. 7].

We say f is an *α-strategy* if $|f(B)| \geq \alpha|B|$ for all B.

Transport. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a k-quasisymmetric map. For each topological disk $D \subset \mathbb{R}^n$, choose a round disk $D^\circ \subset D$ of maximal diameter. Note that property (1) above implies

$$|\phi(B)^\circ| \asymp |\phi(B)|$$

for any ball B, and similarly for ϕ^{-1}.

Using ϕ, we can transport a given positional strategy $f(B)$ to a new strategy $f'(B')$ by defining

$$f'(B') = \phi(f(\phi^{-1}(B')^\circ))^\circ. \tag{2.2}$$

Lemma 2.3 *If f is an α-strategy then f' is an α'-strategy, where α' depends only on (k, n, α).*

Proof. Let $B = \phi^{-1}(B')^\circ$, let $A = f(B) \subset B$, and let $A' = \phi(A)^\circ = f'(B')$. Then by the above properties of ϕ we have

$$\frac{|A'|}{|B'|} \asymp \frac{|\phi(A)|}{|\phi(B)|} \geq \xi_{k,n}\left(\frac{|A|}{|B|}\right) = \xi_{k,n}(\alpha) > 0.$$

Thus we can take $\alpha' \asymp \xi_{k,n}(\alpha)$. ■

Proof of Theorem 2.1 Let $E' = \phi(E)$, and consider $\beta' \in (0,1)$.

Let $f(B)$ denote an (α, β)-strong winning positional strategy for E, where β is yet to be determined. Let $f'(B')$ be the transported strategy defined by (2.2). By the preceding Lemma, f' is an $\alpha' = \alpha'(k, n, \alpha)$ strategy.

Let $B'_1, B'_2, \ldots$ denote a sequence of moves by player B, and let $A'_i = f'(B_i)$. We then have

$$|A'_i| \geq \alpha' |B'_i| \quad \text{and} \quad |B'_{i+1}| \geq \beta' |A'_i|.$$

Let $B_i = \phi^{-1}(B'_i)^\circ$ and let $A_i = f(B_i)$, so $A'_i = \phi(A_i)^\circ$. Then $|A_i| \geq \alpha |B_i|$ for all i. Moreover, there exists a $\beta > 0$ such that

$$|B_{i+1}| \geq \beta |A_i| \tag{2.3}$$

for all i. Indeed, equation (2.1) implies

$$\frac{|B_{i+1}|}{|A_i|} \asymp \frac{|\phi^{-1}(B'_{i+1})|}{|\phi^{-1}(A'_i)|} \geq \xi_{k,n}\left(\frac{|B'_{i+1}|}{|A'_i|}\right) = \xi_{k,n}(\beta') > 0,$$

so (2.3) holds with $\beta \asymp \beta'$.

Since f is (α, β)-strong winning, the intersection $\bigcap A_i = \bigcap B_i$ meets E, and hence $\phi(\bigcap B_i) \subset \bigcap B'_i$ meets $\phi(E) = E'$. Thus f' is a winning strategy for player A. Since $\beta' \in (0,1)$ was arbitrary, this shows E' is an α'-strong winning set. ∎

Proof of Theorem 2.2. A similar argument applies. Let $E \subset \mathbb{R}^n$ be an absolutely winning set. Let $f(B)$ denote a winning positional strategy for player A in the game with rules (1.4) and (1.5), where the constant $\beta > 0$ has yet to be chosen.

Let $E' = \phi(E)$, where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a k-quasisymmetric, map and fix $\beta' > 0$. Consider a sequence of moves $B'_1 \supset B'_2 \supset \cdots$ by player B satisfying $|B'_{i+1}| > \beta' |B_i|$. Let B_i denote the smallest ball containing $\phi^{-1}(B'_i)$; then we have $B_1 \supset B_2 \supset B_3 \cdots$ as well. Let $A_i = f(B_i)$, and define $f'(B'_i)$ to be a ball centered at a point of $\phi(A_i)$ with $|A'_i| = \beta' |B'_i|$.

We claim that for β sufficiently small, the strategy $A'_i = f'(B'_i)$ is absolutely winning for player A. That is, if $B'_{i+1} \subset B'_i - A'_i$ for all i, then $\bigcap B'_i$ must meet E'.

To see this, fix $\epsilon > 0$. We may assume that $|B'_i| \to 0$ (otherwise player A wins, since E' is dense.) Using this assumption, we can replace (A_i, B_i, A'_i, B'_i) with a subsequence such that

$$|B'_{i+1}| < \epsilon |A'_i| \tag{2.4}$$

and yet

$$\inf |B'_{i+1}|/|B'_i| > 0. \tag{2.5}$$

(To achieve this, we ignore player B's responses to the move A_i until the first moment that (2.4) is satisfied.) Also, since $|A_i| \leq \beta |B_i|$ and $|B'_i| \leq (1/\beta')|A'_i|$, if β is small enough then we have

$$|\phi(A_i)| < \epsilon |A'_i|. \tag{2.6}$$

By assumption, B'_{i+1} is disjoint from A'_i, and hence (2.6) implies

$$d(B_{i+1}, \phi(A_i)) > (1/2 - \epsilon)|A'_i|. \tag{2.7}$$

Since ϕ is k-quasisymmetric, there exists an $\epsilon > 0$, depending only on k, such that (2.4) and (2.7) imply that B_{i+1} is disjoint from A_i. But by (2.5), we can still choose $\beta > 0$ so that $|B_{i+1}| \geq \beta |B_i|$ along this subsequence. Then since f is a β-winning strategy, the set $\bigcap B_i$ meets E; and consequently $\bigcap B'_i = \phi(\bigcap B_i)$ meets $E' = \phi(E)$. ■

Remarks. Schmidt's game and its variants can be played in general metric spaces (cf. [Sch]), and Theorems 2.1 and 2.2 hold in this setting as well, provided quasisymmetric maps are defined in such a way that properties (1-4) above hold (they are no longer formal consequences of equation (1.2)). Note that for $n \geq 2$, quasiconformal homeomorphism of $\mathbb{R}^n$ automatically preserve sets of measure zero and sets of dimension n [Vai1, §33], [Geh1, Thm.3]. For more on quasiconformal maps in general metric spaces, see e.g. [TV], [HK].

3 Diophantine sets

In this section we prove the Diophantine set $D(\Gamma)$ is absolutely winning (Theorem 1.3).

Hyperbolic space. Let $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ denote the upper halfspace with coordinates (x, y), endowed with the hyperbolic metric

$$\rho^2 = \frac{|dx|^2 + |dy|^2}{y^2}$$

of constant curvature -1. We will identify $\partial\mathbb{H}^{n+1}$ with the sphere $\mathbb{R}^n \cup \{\infty\}$. For any ball $B = B(x, r) \subset \mathbb{R}^n$, we let

$$\operatorname{top}(B) = (x, r) \in \mathbb{H}^{n+1}.$$

The top of B is the highest point on the hyperbolic plane spanned by ∂B. Each $x \in \mathbb{R}^n$ determines a vertical geodesic

$$\gamma_x(t) = (x, e^{-t}) : \mathbb{R} \to \mathbb{H}^{n+1},$$

parameterized by arclength and satisfying $\gamma_x(t) \to x$ as $t \to \infty$.

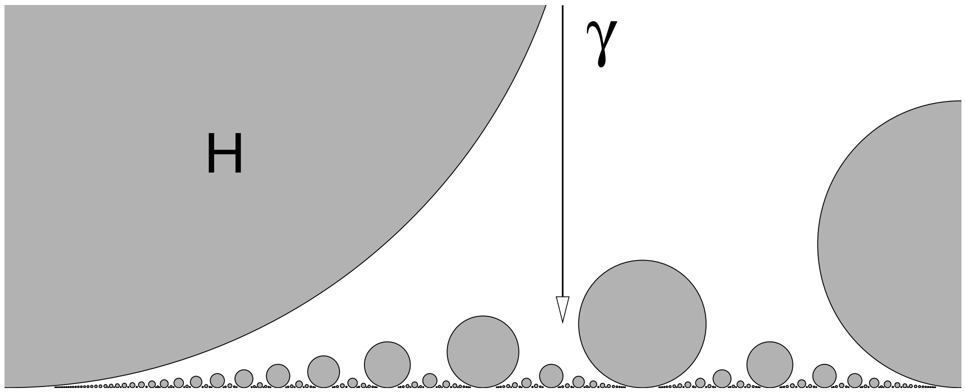

Figure 1. A bounded geodesic avoids an infinite array of horoballs.

Bounded geodesics. Now let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ be a lattice with a cusp at infinity, and let

$$\pi : \mathbb{H}^{n+1} \to M = \mathbb{H}^{n+1}/\Gamma$$

denote the projection to its finite-volume quotient space. The associated Diophantine set is defined by

$$D(\Gamma) = \{x \in \mathbb{R}^n \ : \ \pi(\gamma_x(t)) \text{ remains bounded as } t \to +\infty\};$$

equivalently, $\overline{\pi(\gamma_x(0,\infty))} \subset M$ is compact.

Horoballs and cusps. A *horoball* $H \subset \mathbb{H}^{n+1}$ is a hyperbolic ball of 'infinite radius', centered at a point $p \in \partial\mathbb{H}^{n+1}$. A horoball centered at $p \in \mathbb{R}^n$ is simply a Euclidean ball of the form

$$H = H(p,r) = B((p,r),r) \subset \mathbb{R}^{n+1}.$$

A horoball centered at $p = \infty$ is a halfspace of the form $H(\infty, r) = \{(x,y) \ : \ y > 1/r\}$. We will use the shorthand ϵH for $H(p, \epsilon r)$ when $H = H(p,r)$. Note that ∂H are $\partial(\epsilon H)$ are parallel in the hyperbolic metric, with separation $|\log \epsilon|$.

The basic structure theorems for finite-volume hyperbolic manifolds imply there is a Γ-invariant collection of disjoint horoballs $\mathcal{H}'$ such that

$$M_\epsilon = \overline{\mathbb{H}^{n+1} - \bigcup\{\epsilon H \ : \ H \in \mathcal{H}'\}}/\Gamma$$

is compact for every $\epsilon > 0$, and $M = \bigcup_{\epsilon>0} M_\epsilon$. There is a unique infinite horoball $H(\infty, r) \in \mathcal{H}'$; we let $\mathcal{H} \subset \mathcal{H}'$ denote the remaining set of finite horoballs. The Diophantine set $D(\Gamma)$ can then be described as follows:

A point $x \in \mathbb{R}^n$ belongs to $D(\Gamma)$ $\iff$ There is an $\epsilon > 0$ such that $\gamma_x(t)$ avoids ϵH for all $H \in \mathcal{H}$ and $t > 0$.

See Figure 1.

Porosity: Proof of Proposition 1.5. By compactness of M_ϵ, the set $D_\epsilon(\Gamma)$ where $\gamma_x(t)$ avoids $\bigcup \epsilon H$ is porous; hence $D(\Gamma) = \bigcup D_{1/n}(\Gamma)$ is σ-porous. ■

The game. Let us now analyze a typical game

$$B_1 \supset (B_1 - A_1) \supset B_2 \supset (B_2 - A_2) \supset B_3 \cdots$$

played using the rule

$$|A_i| \leq \beta |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |B_i|$$

for a fixed $\beta \in (0, 1/3)$. Suppose for simplicity that $\bigcap B_i$ consists of a single point x. Let $B_i = B(x_i, r_i)$ and let $t_i = -\log r_i$. Since $|x - x_i| \leq r_i$, the points $\mathrm{top}(B_i) = (x_, r_i)$ and points $\gamma_x(t_i) = (x, r_i)$ satisfy

$$d(\gamma_x(t_i), \mathrm{top}(B_i)) \leq 1 \tag{3.1}$$

in the hyperbolic metric. We also have

$$|t_i - t_{i+1}| = \log(|B_i|/|B_{i+1}|) \leq |\log \beta|. \tag{3.2}$$

Thus the sequence of points $\mathrm{top}(B_i)$ is a good approximation to the geodesic γ_x. In particular, these points record the horocyclic excursions of γ_x.

Lemma 3.1 *Suppose $H = H(p, r) \in \mathcal{H}$ and $\gamma_x(t) \in \epsilon H$ for some $t > t_1$, where $\epsilon = \beta/e$. Then there is an $i \geq 1$ such that $\mathrm{top}(B_i) \in H$. Moreover, i can be chosen so that either $|B_i| \geq \epsilon r$ or $i = 1$.*

Proof. Let $H' = (1/e)H$, and let $\gamma_x^{-1}(H') = (a, b) \subset \mathbb{R}$. Then for any $i \geq 1$,

$$t_i \in (a, b) \implies \mathrm{top}(B_i) \in H,$$

since $d(\partial H, \partial H') = 1$ and $d(\gamma_x(t_i), \mathrm{top}(B_i)) \leq 1$. We also have

$$|a - b| \geq d(\partial H', \partial(\epsilon H)) = |\log \beta|,$$

since the path $\gamma_x(a, b)$ joins $\partial H'$ to $\partial(\epsilon H)$.

Since $t_1 < t \in [a, b]$, we have $b > t_1$. If $t_1 \geq a$ then $\mathrm{top}(B_1) \in H$, and the proof is complete.

Otherwise $t_1 < a$. Since $t_i \to \infty$ with $|t_i - t_{i+1}| \leq b - a$ (by equation (3.2), there exists an i such that $t_i \in [a, b]$. This implies $\mathrm{top}(B_i) \in H$. Moreover, we have $t_i - a \leq |\log \beta|$. Since a vertical geodesic meeting

H' must cross the horizontal diameter of H', we also have $r/e \le e^{-a}$ and hence

$$\epsilon r = \beta r/e \le \beta e^{-a} \le e^{-t_i} = r_i = |B_i|$$

as desired. ■

The strategy. Now fix $\beta \in (0, 1/3)$. We define a positional strategy $A = f(B) \subset B$ as follows:

1. If $\mathrm{top}(B) \in H = H(p, r) \in \mathcal{H}$, then we choose $A = B(p, s)$ with $|A| = 2s = \beta|B|$.
2. Otherwise, we choose A disjoint from B.

It is immediate that this strategy satisfies:

Lemma 3.2 *Whenever $x \in B - f(B)$ and $\mathrm{top}(B)$ meets $H = H(p, r) \in \mathcal{H}$, the geodesic γ_x avoids $H(p, s)$ where $2s = \beta|B|$.*

Theorem 1.3 now follows from:

Theorem 3.3 *The strategy f is absolutely winning for the set $D(\Gamma)$ with constant β.*

Proof. Consider any game in which player A's moves are given by $A_i = f(B_i)$, subject to the rules $|B_{i+1}| \ge \beta|B_i|$ and $B_{i+1} \subset B_i - A_i$ for player B. We may assume $\bigcap B_i = \{x\}$ is a single point (otherwise player A wins because $D(\Gamma)$ is dense). To complete the proof, we will show there is a $\delta > 0$ such that $\gamma_x(t)$ avoids $\bigcup_{\mathcal{H}} \delta H$ for all $t > t_1$. (This is equivalent to showing $x \in D(\Gamma)$.)

Let $\epsilon = \beta/e$, and let

$$\mathcal{H}_x = \{H \in \mathcal{H} \,:\, \gamma_x(t) \in \epsilon H \text{ for some } t > t_1\}.$$

By Lemma 3.1, for each $H \in \mathcal{H}_x$ there is an $i \ge 1$ such that $\mathrm{top}(B_i) \in H = H(p, r)$. Let us use the least such i to label $H = H_i = H_i(p_i, r_i)$, and let I denote the set of indices so obtained; then $\mathcal{H}_x = \{H_i : i \in I\}$. By Lemma 3.2, γ_x avoids $\bigcup \delta_i H_i$, where $\delta_i = (1/2)\beta|B_i|/r_i$. So if we set $\delta = \min(\epsilon, \inf_I \delta_i)$, then $\gamma_x(t)$ avoids $\bigcup_{\mathcal{H}} \delta H$ for all $t > t_1$. Note that Lemma 3.1 also gives $|B_i|/r_i \ge \epsilon > 0$ for all $i \in I$ except possibly $i = 1$; thus $\inf_I \delta_i > 0$, and hence $\delta > 0$, completing the proof. ■

Remark: The steering game. Here is a related game which can be played in hyperbolic space. Two players take turns choosing smooth segments in hyperbolic space satisfying

$$|A_i| \le a \quad \text{and} \quad |B_i| \le b$$

for fixed $a, b > 0$, such that the segments fit together to form a smooth ray

$$\gamma = B_1 \cup A_1 \cup B_2 \cup A_2 \ldots \subset \mathbb{H}^{n+1}$$

with geodesic curvature $< 1/2$. Then γ is a quasigeodesic, so it converges to a unique point $x \in \partial\mathbb{H}^{n+1}$ (provided its length is infinite). We say $E \subset \mathbb{R}^n$ is winning for the *steering game* if there is an $a > 0$ such that for all $b > 0$, player A can insure that $x \in E$.

It is straightforward to show that E is winning for the steering game iff E is strong winning. Thus Theorem 1.3 has the following corollary: if A is driving a car at (high) uniform speed in a finite-volume hyperbolic manifold M, but he can only control the wheel for one minute every hour, then he can still keep the car within a bounded distance from home (provided the car makes no sharp turns.)

We note that the bounded geodesic $\pi(\gamma_x) \subset M$ produced by player A using the strategy of Theorem 3.3 lies in a set of diameter $O(|\log\beta|)$ (apart from some initial excursions into the cusps which can be forced by B's first move).

4 Counterexamples

Given $x \in \mathbb{R}$, we will write the fractional part of x in base 2 as $\{x\} = 0.x_1x_2x_3\ldots$, and set $Z(x) = \{s \,:\, x_s = 0\}$. Let

$$W = \{x \in \mathbb{R} \,:\, Z(x) \text{ contains an almost arithmetic progression}\}.$$

Here an *almost arithmetic progression* is a set of the form

$$P = \lfloor \sigma + \mathbb{N}\theta \rfloor \subset \mathbb{Z},$$

where $\sigma, \theta \in \mathbb{R}$ and $\theta > 0$.

In this section we will show:

Theorem 4.1 *The set W is winning but not strong winning.*

Theorem 4.2 *For every $k > 1$ there is a k-quasisymmetric map $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(W)$ is not winning.*

Winning. We begin the proof of Theorem 4.1 by showing:

Proposition 4.3 *The set W is α-winning for all $\alpha < 1/8$.*

Proof. Note that $|B_{i+1}| = (\alpha\beta)^i|B_1|$ for all $i > 0$. Choose σ, θ so that $2^{-\theta} = \alpha\beta$ and $2^{-\sigma-\theta} = (1/4)|B_1|$; then we have

$$2^{-\sigma-i\theta} = (1/4)|B_i|$$

for all i. Thus the integers $s_i = \lfloor \sigma + i\theta \rfloor$ form an almost arithmetic progression that satisfies

$$(1/2)|B_i| \geq 2^{-s_i} \geq (1/4)|B_i| > 2|A_i|$$

for all i. These bounds imply player A can choose A_i such that $x_{s_i} = 0$ for all $x \in A_i$. Then the unique point $x \in \bigcap A_i$ will belong to W, since $\{s_1, s_2, \ldots\} \subset Z(x)$. ∎

To show W is not strong winning, we first note:

Lemma 4.4 *Suppose $0 < \beta < \alpha^{100}$. Then in the (α, β)-Schmidt game, player B can force the unique point $x \in \bigcap B_i$ to satisfy*

$$Z(x) \subset Z_\theta = B(\lfloor \mathbb{N}\theta \rfloor, \theta/10),$$

where $\theta = -\log_2(\alpha\beta)$.

Proof. We may assume $\alpha < 1/2$, since otherwise player B has complete control over x. Suppose player B uses the following strategy: he begins the game by choosing $B_1 = [0, 1]$, and thereafter chooses B_i to force $x_j = 1$ as often as possible. Then, since $|B_{i+1}| = (\alpha\beta)^i = 2^{-i\theta}$, while $|A_{i+1}| = \alpha|B_{i+1}| > 2^{-i\theta-\theta/20}$, player A can only control the digit x_j if $j \in Z_\theta$, and player B's strategy forces $x_j = 1$ for all remaining j. ∎

Lemma 4.5 *Suppose $P = \lfloor \sigma + \mathbb{N}\theta \rfloor$ is close to $P' = \lfloor \sigma' + \mathbb{N}\theta' \rfloor$, in the sense that*

$$P \subset B(P', \theta'/6 - 2).$$

Then $\theta \in \mathbb{N}\theta'$.

Proof. The hypotheses imply that

$$\mathbb{N}(\theta/\theta') \subset B(\mathbb{N} + \delta, 1/6),$$

where $\delta = (\sigma' - \sigma)/\theta'$. Hence $G = \mathbb{N}(\theta/\theta') \bmod 1$ is contained in an interval of length less than $< 1/3$ on $S^1 = \mathbb{R}/\mathbb{Z}$. But there is no nontrivial semigroup contained in such a short interval, so $G = \{0\}$ and $\theta/\theta' \in \mathbb{Z}$. ∎

Proof of Theorem 4.1. We will show that for each $\alpha \in (0, 1)$, the set W is not strong winning for $\beta = \alpha^{200}$. We may assume $\alpha < 1/2$.

First note that, upon replacing β with β/α, we can assume $|A_i| = \alpha|B_i|$ for every i. If player A does not choose $|A_i|$ this small, player B can choose an arbitrary $A'_i \subset A_i$ with $|A'_i| = \alpha|B_i|$ and then respond to this move instead.

Now for any $\beta' \in [\alpha^{100}, \alpha^{101}]$, the moves for the (α, β')-Schmidt game are also legal for the modified (α, β) game defined by (1.3). Thus player B can follow the strategy provided by Lemma 4.4 to insure that

$$Z(x) \subset B(\lfloor \mathbb{N}\theta' \rfloor, \theta'/10),$$

where $\theta' = -\log_2(\alpha\beta') > 100$. Thus if $Z(x)$ contains an almost arithmetic progression $\lfloor \sigma + \mathbb{N}\theta \rfloor$, Lemma 4.5 guarantees that

$$\theta \in \mathbb{N}\theta' \tag{4.1}$$

(since $\theta'/10 < \theta'/6 - 2$).

Player B has a similar strategy that guarantees

$$\theta \in \mathbb{N}\theta'', \tag{4.2}$$

for any θ'' close enough to θ'. By alternating between longer and longer runs of each of these two strategies, player B can insure that both (4.1) and (4.2) hold. To win the game, player B simply chooses β' and β'' such that θ''/θ' is irrational. Then $(\mathbb{N}\theta') \cap (\mathbb{N}\theta'') = \{0\}$; therefore $Z(x)$ contains no almost arithmetic progression, and hence $x \notin W$. ■

Quasisymmetric maps. We now turn to the proof of Theorem 4.2. The idea of the proof is that, by applying a quasisymmetric map to W, we can essentially allow player B to vary the value of β. Since W is not strong winning, ϕ can be chosen so $\phi(W)$ is not winning.

To get some breathing room, we will write $W = \bigcup_{T=1}^{\infty} W_T$, where

$$W_T = \{x \in \mathbb{R} \ : \ Z(x) \supset \lfloor \sigma + \mathbb{N}\theta \rfloor \text{ for some } \sigma, \theta \in [1, T] \ \}.$$

It is easy to see that $W_T \cap [0,1]$ is a closed, porous set, and thus:

Proposition 4.6 *The winning set W is a σ-porous set.*

The following tool will be useful in the construction of ϕ.

Lemma 4.7 *Let $E \subset [0,1]$ be a porous set. Then there is a family of $k(u)$-quasisymmetric maps $\phi_u : [0,1] \to [0,1]$ defined on a neighborhood of $u = 1$, such that*

$$(1/2)|B|^u < |\phi_u(B)| < 2|B|^u$$

for all balls $B \subset [0,1]$ meeting E, and $k(u) \to 1$ as $u \to 1$.

Idea of the proof. One can construct $\phi_u = \cdots \phi_3 \circ \phi_2 \circ \phi_1$ by composing an infinite sequence of smooth quasisymmetric maps where $\phi_i'(x) = 2^{1-u}$ at scale 2^{-i} near E, and ϕ_i is nearly the identity at larger scales. By taking advantage of the complementary gaps (provided by porosity), one can insure that the composition is quasisymmetric with $k(u)$ close to 1. ■

The existence of such a ϕ is not surprising, since a k-quasisymmetric map need only be $(1/k)$-Hölder continuous. Such maps arise naturally as conjugacies in 1-dimensional dynamics.

Shifting strategies. Now fix $\alpha' \in (0, 1/2)$ and set $\beta' = (\alpha')^{200}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a k-quasisymmetric map. Consider a sequence of moves (A_i', B_i') for the (α', β')-Schmidt game, with x' the unique point in $\bigcap A_i'$. The preimages of A_i', B_i' and x' under ϕ will be denoted by A_i, B_i and x.

We will present a sequence of results leading up to a construction of a mapping ϕ such that $W' = \phi(W)$ is not (α', β')-winning.

The strategy for player B will always be the same; we call it the *basic strategy*. First, player B chooses B_1' so that $B_1 = [0, 1]$. Then, for each $i > 1$, player B chooses B_i' so that $x_j = 1$ for all $x \in B_i$ and all j in the range he can control. This range is essentially

$$J_i = \{j \ : \ -\log_2 |A_{i-1}| < j < -\log_2 |B_i|\}.$$

A key point is that J_i depends on ϕ; we have $|B_i'| = (\alpha\beta)^{i-1}$, but $|B_i|/|B_i'|$ may be large or small. On the other hand, if k is close to 1, then $|B_i|$ is close to $\beta'|A_{i-1}|$, so the *number* of j that player B can control with a single move (the length of J_i) is still close to $-\log_2 \beta'$.

Let $\theta' = -\log_2(\alpha'\beta')$.

Lemma 4.8 *Fix T and $k > 1$. Then for θ close enough to θ', there is a k-quasisymmetric map ϕ such that if B plays the basic strategy, and $x \in W_T \cap \bigcap B_i$, then*

$$Z(x) \subset B(\lfloor \mathbb{N}\theta \rfloor, \theta/10). \tag{4.3}$$

Proof. Observe that the strategy for B employed in the proof of Lemma 4.4 is simply the basic strategy in the case $\phi(x) = x$. Thus Lemma 4.4 gives the result above in the special case $\theta = \theta'$.

For the general case, let $u = \theta'/\theta$, and apply Lemma 4.7 to construct a map $\phi : \mathbb{R} \to \mathbb{R}$ with the property that $|A_i|^u \asymp |A_i'|$ and $|B_i|^u \asymp |B_i'|$ whenever $x \in W_T$. This means (A_i, B_i) are essentially moves in the (α, β)-Schmidt game, where $(\alpha^u, \beta^u) = (\alpha', \beta')$. The basic strategy for player B then produces a sequence of moves B_i of the same type as was used in the proof of Lemma 4.4; and this insures the inclusion (4.3) holds whenever $x \in W_T$. ∎

Lemma 4.9 *For all T and $k > 1$, there exists a k-quasisymmetric ϕ such that $\phi(W_T)$ is not (α', β')-winning.*

Proof. To construct ϕ, choose θ'' close to θ' such that their ratio is irrational. Then there is an $r > 0$ such that

$$B(\lfloor \mathbb{N}\theta' \rfloor, r) \cap B(\lfloor \mathbb{N}\theta'' \rfloor, r) \cap [0, T] = \emptyset. \tag{4.4}$$

Let $u = \theta''/\theta'$. Choose a large integer N and, using Lemma 4.7, construct a k-quasisymmetric mapping ϕ such that

1. $\phi(x)_j = x_j$ for all $j \leq N$; and
2. $(2^N|\phi(B)|)^u \asymp (2^N|B|)$ for any ball B meeting W_T with $2^{-2N} < |B| < 2^{-N}$.

Since u is close to 1, we can also arrange that k is close to 1, independent of N.

Let $Z_N(x) = Z(x) \cap [0, N]$. Because of (1), the basic strategy for player B forces

$$Z_N(x) \subset B(\lfloor \mathbb{N}\theta' \rfloor, \theta'/10).$$

On the other hand, because of (2), it also forces

$$Z_{2N}(x) \backslash Z_N(x) \subset B(\lfloor \mathbb{N}\theta'' \rfloor, \theta''/10).$$

Now suppose $x \in W_T$. Then $Z(x) \supset \lfloor \sigma + \mathbb{N}\theta \rfloor$ for some $\sigma, \theta \leq T$. For N sufficiently large, the two conditions above insure that θ is within distance r of both $\lfloor \mathbb{N}\theta' \rfloor$ and $\lfloor \mathbb{N}\theta'' \rfloor$. But there is no such θ, by equation (4.4).

Thus player B wins if he uses the basic strategy; indeed, his win is guaranteed by the time he chooses B_{2N}. ■

Final jeopardy: Proof of Theorem 4.2. Given $k > 1$, we will construct a k-quasisymmetric map such that $W' = \phi(W)$ is not winning. The map will be normalized so that $\phi[0, 1] = [0, 1]$.

Player B will use the basic strategy. This strategy can be carried out so there are only finitely many choices for each B_i , and each choice satisfies

$$W \cap \partial B_i = \emptyset.$$

This last condition is easy to achieve since W is nowhere dense.

Along with ϕ we will construct a sequence of disjoint intervals $N_{n,T} \subset \mathbb{N}$, indexed by integers $n, T > 0$, such that whenever B plays the basic strategy for the (α', β')-game, we have:

$$(\alpha', \beta') = (2^{-n}, 2^{-200n}) \implies W_T \cap \bigcap_{i \in N_{n,T}} B_i = \emptyset. \tag{4.5}$$

This suffices to complete the proof. Indeed, if $\phi(W)$ is winning, then it is $(2^{-n}, 2^{-200n})$-winning for some $n > 0$ (cf. [Sch, Lemma 11]); but (4.5) implies that, if B follows the basic strategy, then the unique point $x \in \bigcap B_i$ will not belong to $\bigcup W_T = W$.

To construct ϕ, let (n_j, T_j), $j = 1, 2, 3, \ldots$ enumerate all the possible pairs (n, T) as above. Let $(\alpha_j, \beta_j) = (2^{-n_j}, 2^{-200n_j})$, and choose $k_j > 1$ such that $\prod k_j = k$.

By Lemma 4.9, there is a k_1-quasisymmetric map such that $\phi_1(W_{T_1})$ is not (α_1, β_1)-winning. In fact (as we have seen in the proof), ϕ_1 can be chosen so that player B wins after a finite set of moves, ranging in a finite interval $N_{n_1,T_1} \subset \mathbb{N}$.

Then $\phi = \phi_1$ yields (4.5) for $(n, T) = (n_1, T_1)$. Let $F_1 \subset [0, 1]$ be the finite set of possible endpoints for B_i with $i \in N_{n_1,T_1}$.

Since F_1 is finite, we may also arrange that ϕ_1 is differentiable; it can be smoothed at a small scale without changing the values of $\phi(x), x \in F_1$, and hence without changing (4.5) for (n_1, T_1).

By differentiability, the compact set $\phi_1(W_{T_2}) \cap [0, 1]$ can be covered by a collection of disjoint intervals $\mathcal{I}_2$ which each meet $\phi_1(W_{T_2})$ in a nearly linearly rescaled piece of W_{T_2}. Since $\phi_1(F_1)$ is disjoint from $\phi_1(W) \supset \phi_1(W_{T_2})$, we can also assume that $d(I, \phi_1(F_1)) \gg |I|$ for each $I \in \mathcal{I}_2$.

By applying Lemma 4.9 in each of these intervals and assembling the results, we obtain a k_2-quasisymmetric mapping ϕ_2 such that $W_2' = \phi_2(\phi_1(W_{T_2}))$ is not (α_2, β_2)-winning. Moreover, we can find an interval N_{n_2,T_2} disjoint from N_{n_1,T_2} such that player B's moves in this range already exclude x from W_{T_2}. Finally, we can arrange that $\phi_2(x) = x$ for all $x \in \phi_1(F_1)$.

Then $\phi = \phi_2 \circ \phi_1$ yields (4.5) for both $(n, T) = (n_1, T_1)$ and $(n, T) = (n_2, T_2)$. Continuing in the same fashion, we obtain in the limit a mapping

$$\phi = \lim_{j\to\infty} \phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1$$

such that (4.5) holds for all $n, T \geq 1$. The limit is k-quasisymmetric because $\prod k_j = k$. ■

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